Higher-order differential equations

Peyam Tabrizian

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This handout is meant to give you a couple more example of all the techniques discussed in chapter 6, to counterbalance all the dry theory and complicated applications in the differential equations book! Enjoy! :)

1 Homogeneous equations

1.1 Solve $y''' + y'' - 2y = 0$

Method: Find the aux. equation, factor it out, and find the solution.

Aux. equation: $p(r) = r^3 + r^2 - 2 = 0$

To factor this, use the following theorem:

Rational Roots Theorem If $r = \frac{a}{b}$ is a zero of p, then a divides -2 (the constant term of p), and b divides 1 (the leading coeff. of p).

This gives $a = \pm 1, \pm 2$ (those are the only numbers which divide -2) and $b = \pm 1$ (the only numbers which divide 1), so $r = \frac{a}{b} = \pm 1, \pm 2$.

This means: plug in $r = 1, -1, 2, -2$ into your aux. equation, and **STOP** once you find a zero of p.

Tip: Always start with the easiest guess!

Here $p(1) = 1 + 1 - 2 = 0$, so $r = 1$ works!

Now use long division (divide $r^3 + r^2 - 2$ by $r - 1$) to get:

$$p(r) = (r - 1)(r^2 + 2r + 2) = 0$$

Which gives $r = 1$ and $r = 1 \pm i$ (quadratic formula)

Hence the general solution is:

$$y(t) = Ae^t + Be^t \cos(t) + Ce^t \sin(t)$$

Note: You might have to do this process (rat. roots thm and long division) several times, especially if you deal with fourth-degree polynomials)

Note: Sometimes there are shortcuts to this. For example, $r^4 + 4r^2 + 4 = (r^2 + 2)^2$, and in this case you don't have to use any long-division.

1.2 Find a general solution to $(D - 1)^3(D^2 + 2D + 5)^2[y] = 0$

Aux: $(r - 1)^3(r^2 + 2r + 5)^2 = 0$ (replace D by r)

This gives $r = 1$ (multiplicity 3) and $r = -2 \pm i$ (multiplicity 2), hence:

$$y(t) = Ae^t+Bte^t+Ct^2e^t+De^{-2t}\cos(t)+Ee^{-2t}\sin(t)+Fte^{-2t}\cos(t)+Gte^{-2t}\sin(t)$$

Note: This is **exactly** why we asked you the problem in this form, so that the auxiliary polynomial is easy to factor out.

2 Intervals of existence

2.1 Find the largest interval (a, b) on which the following differential equation has a unique solution:

$$(t-4)y'' + \sqrt{t^2-1}y' + y = \ln(t)$$

with

$$y(2) = 0, y'(2) = 5$$

IMPORTANT: First divide the equation by the leading term:

$$y'' + \left(\frac{\sqrt{t^2-1}}{t-4}\right)y' + \left(\frac{1}{t-4}\right)y = \frac{\ln(3-t)}{t-4}$$

Now look at the domain of each term, including the inhomogeneous term:

$\underline{y'-\text{term}}$: The domain of $\frac{\sqrt{t^2-1}}{t-4}$ is $(-\infty, -1] \cup [1, 4) \cup (4, \infty)$.

However, because $(-\infty, -1)$ and $(4, \infty)$ do **not** contain the initial condition 2, we ignore them and only consider the interval $\boxed{[1, 4)}$

$\underline{y-\text{term}}$: The domain of $\frac{1}{t-4}$ is $(-\infty, 4) \cup (4, \infty)$, and we only consider $\boxed{(-\infty, 4)}$

<u>Inhom. term</u>: The domain of $\frac{\ln(3-t)}{t-4}$ is $\boxed{(-\infty, 3)}$

(remember that the domain of $\ln(t)$ is $(0, \infty)$).

Finally **intersect** the intervals to get: $[1, 4) \cap (-\infty, 4) \cap (-\infty, 3) = [1, 3)$. (draw a picture if necessary). Note that this interval indeed contains the initial condition 2

And because we want an *open* interval (a, b), the answer is: $\boxed{(1, 3)}$ (this is just a technicality)

3 Linear independence

3.1 Are the functions $\cos^2(x), \sin^2(x), 1$ linearly dependent or independent?

Linearly dependent because $\cos^2(x) + \sin^2(x) = 1$.

Point: Linear *dependence* is usually easier to check than linear *independence*! That's why for the rest we're only going to focus on linear independence.

3.2 Determine if $f(t) = \cos(t)$ and $g(t) = \sin(t)$ are linearly dependent or independent

Form the (pre)-Wronskian:

$$\widetilde{W}(t) = \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

Notice how you build the (pre)-Wronskian: You put all the functions on the first row, and you differentiate as many times until you get a square matrix. This also works for more than 2 functions.

Now pick your favorite point t and evaluate the determinant of the above matrix **at that point**. For example, pick $t = 0$:

$$det(\widetilde{W}(0)) = det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

Since the determinant is $\neq 0$, f and g are linearly independent.

Note: If you find that the det is $= 0$, do **NOT** conclude that the functions are linearly dependent! See next example!

3.3 Are the functions $f(t) = t, g(t) = t^2, h(t) = t^3$ linearly independent or depedent?

$$\widetilde{W}(t) = \begin{bmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{bmatrix} = \begin{bmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{bmatrix}$$

Now pick $t = 1$ (see Note below), then:

$$det\left(\widetilde{W}(1)\right) = det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{bmatrix} = 2 \neq 0$$

Hence t, t^2, t^3 are linearly independent.

Note: Had you picked $t = 0$, you would have found that the determinant is $= 0$. However, this does **not** mean that the functions are linearly dependent. Just continue choosing points until you find that the functions are linearly independent. For example, $t = 1$ works!

3.4 Determine if $te^t, t^2e^t, t^3e^t, t^4e^t$ are linearly independent

Before you tackle the Wronskian, always see if you can simplify your functions a bit! For this, use the definition of linear independence: Suppose that

$$Ate^t + Bt^2e^t + Ct^3e^t + Dt^4e^t = 0$$

Now cancel out e^t and you get:

$$At + Bt^2 + Ct^3 + Dt^4 = 0$$

Furthermore, **provided** $t \neq 0$ (the 'illegal value', we don't want to divide by 0) cancel out the t:

$$A + Bt + Ct^2 + Dt^3 = 0$$

So you only have to check if $1, t, t^2, t^3$ are linearly independent!

Now use the (pre)-Wronskian:

$$\widetilde{W}(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 2 & 6t \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Pick $t = 1$, which is good because $t \neq 0$ (the illegal value):

$$det(\widetilde{W}(1)) = det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 12 \neq 0$$

Hence the functions are <u>linearly independent</u>.